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Book Reviews

Books

H. STAHL AND V. TOTIK, *General Orthogonal Polynomials*, Encyclopedia of Mathematics and Its Applications, Vol. 43, Cambridge University Press, 1992, xii + 250 pp.

In the present book the authors consider general orthogonal polynomials in the sense that the orthogonality is with respect to a positive Borel measure μ in the complex plane. The emphasis is on the asymptotic behavior of $|p_n(z;\mu)|^{1/n}$ as $n \to \infty$. Such asymptotic behavior is relevant for finding the domain of convergence of Fourier series in orthogonal polynomials. Taylor series converge in disks whereas Legendre (Jacobi) series converge in elliptic domains. Under reasonable assumptions on μ and its support $S(\mu)$ one expects convergence of the Fourier series in a domain of the form $\{g_\Omega(z) < C\}$, where $g_\Omega(z)$ is the Green function of the unbounded component Ω of the support $S(\mu)$. For this nth root asymptotic behavior the natural theoretical setting is (logarithmic) potential theory and this book is a great demonstration of mathematical analysis in general and potential theory in particular. Already on page 4 we find one of the most beautiful results of the last few years, namely the upper and lower bounds for $|p_n(z;\mu)|^{1/n}$. The lower bound $e^{g_\Omega(z)}$ is indeed in terms of the Green function for the outer component Ω of the support $S(\mu)$, but in general a possibly larger upper bound $e^{g_\mu(z)}$ is valid, which is related to the carriers of μ. Carriers are Borel sets B such that $\mu(B) = \mu(S(\mu))$. The support is a carrier, but quite often it is too big for various estimates. Carriers give more detail on how the mass of μ is distributed over its support.

The first two chapters deal with general properties of orthogonal polynomials, bounds, norms, and inequalities in terms of capacities of supports and carriers. Chapters 3, 4, and 5 deal with the class of regular measures, i.e., measures μ such that the nth root asymptotic behavior is determined by the equilibrium measure of the support and the Green function of the unbounded component of the support. In Chapter 4, for instance, various sufficient conditions are given for a measure to be regular, and the relation between known conditions by Erdős-Turán, Ullman, and Widom is clarified. Chapter 6 deals with seven applications such as rational interpolation to Markov functions, best rational approximation to Markov functions with a remarkable new result on the exact rate of convergence, convergence of non-diagonal Padé approximants to Markov functions, and weighted polynomials in L^p.

This is an important but difficult book, necessary to anybody interested in advanced features of complex approximation theory, orthogonal polynomials, measure theory, and logarithmic potential theory. An introduction in (logarithmic) potential theory is recommended and readers are advised to consult, at an early stage, the appendix in which a brief survey is given of the most relevant aspects of potential theory.

ALPHONSE MAGNUS AND WALTER VAN ASSCHE

L. LORENTZEN AND H. WAADELAND, *Continued Fractions with Applications*, Studies in Computational Mathematics, Vol. 3, North-Holland, 1992, xvi + 606 pp.

Continued fractions have always played an important role in mathematics, sometimes in disguise as for instance in the theory of difference equations (although one could always argue that continued fractions are but another face of this theory). They have always acted as a binding factor between diverse subjects such as orthogonal polynomials, geometry of zeros,

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moment problems, and stability. And now there is finally a book on continued fractions that touches on many of the different facets and moreover is quite suitable as a book to use for the uninitiated. That is probably the most important property of the book under review; it can be used to lead young students to the very brink of the unknown while—during the whole process—keeping them interested and showing them the beauty of mathematics. The book is not cheap (not yet: a student edition in paperback format might lower the price considerably), but it is a must for every library. It can not only be used to make students enthusiastic about the subject, but it will serve a far greater goal; it will contribute to the "dusting off" of mathematics and help show that doing mathematics can easily compete with the great experimental sciences regarding beauty, depth of concepts from simple starting points, and joy for the people involved.

The book consists of 12 chapters, an appendix containing a wealth of standard continued fraction expansions (elementary functions, hypergeometric functions, basic hypergeometric functions), and a subject index. Each of the chapters has its own list of references and its own set of problems (this is really extremely well done; the problems turn the natural curiosity of the reader into eagerness to start working). Nearly all chapters are followed by remarks pointing out new directions and ongoing research. A short overview of the 12 chapters follows.

1. *Introductory examples* (54 pages): A very nice introduction with examples, connection with one or two power series, some classical convergence theorems, and general remarks on modification and different types of convergence. Really whets the appetite;

2. *More basics* (38 pages): The concept of "tails" (right and wrong ones), speed of convergence, truncation error, and transformation of continued fractions (including contraction and extension);

3. *Convergence criteria* (96 pages): This chapter exhibits the different tools to be used in the convergence business: linear fractional transformations, convergence-sets, value-sets (leading to parabola and oval theorems), limit-periodicity;

4. *Continued fractions and three-term recurrence relations* (54 pages): Here the intimate connection with the solution-space of three-term recurrence relations is looked into, leading to a.o. the equivalence between existence of a minimal solution and convergence;

5. *Correspondence of continued fractions* (51 pages): The viewpoint is now that of rational approximation in the normed field of formal power series. At the end of the chapter some attention is paid to *branched continued fractions* (introduced by V. Ya. Skorobogatko);

6. *Hypergeometric functions* (40 pages): This concerns the well-known ${}_2F_1$ and the confluent hypergeometric functions, along with the basic hypergeometric functions ${}_2\varphi_1(a, b; c; q; z)$;

7. *Moments and orthogonality* (36 pages): The chapter goes into moment functionals, the Stieltjes moment problem, Favard's theorem, Jacobi-fractions, and Gaussian quadrature;

8. *Padé approximants* (30 pages): Here the classical theory is treated, followed by a discussion of some generalizations;

9. *Some applications in number theory* (44 pages): The usual connection via the Euclidean algorithm with applications to best approximation, diophantine equations, and factorization of integers;

10. *Zero-free regions* (40 pages): Several theorems on zero-free regions are given along with a short discussion and two criteria on stability of polynomials;

11. *Digital filters and continued fractions* (40 pages): An excellent introduction into a subject that has led to cross-fertilization. Key-words are here: the Schur algorithm, stability, and rational transfer function;

12. *Applications to some differential equations* (38 pages): Several examples are given (a.o. Riccati and (almost) Euler-Cauchy) and a very recent method of solving the Riccati equation using *T*-fractions (due to S. C. Cooper) is treated.

MARCEL DE BRUIN

R. A. LORENTZ, *Multivariate Birkhoff Interpolation*, Lecture Notes in Mathematics, Vol. 1516, Springer-Verlag, 1992, ix + 192 pp.

Multivariate polynomial interpolation is a subject which always has attracted much attention, because it is basic in many other mathematical problems: finite elements, splines, cubature formulas, etc. Many papers and theses have been written on this subject in the last decade, but a clear convincing theory, accessible to practitioners, is yet to be achieved. Due to this, most of the textbooks on numerical analysis or approximation theory omit the problem or include only a few pages on it. There is a lack of books entirely devoted to the subject. The present book under review can be considered as a natural continuation of the book *Birkhoff interpolation* by G. G. Lorentz, K. Jetter, and S. Riemenschneider (Addison–Wesley, 1983), which deals with univariate Birkhoff interpolation problems, that is problems with derivatives of any order as interpolation data.

R. A. Lorentz presents his own research on the subject in a good part of the book and some other points of view in the rest, including a long list of references which, if not exhaustive, will be very useful to the reader. The first of the 13 sections is introductory and is followed by 6 others in which the author extends the univariate techniques to multivariate problems. After a section of examples from the theory of finite elements there are two sections (9 and 10) in which the results of the previous sections are applied to Hermite problems, that is problems with the same number of data at every node. In Section 11 several ways of computing Vandermonde-like determinants arising in multivariate interpolation problems are given. The last two sections deal with approaches where the dimension of the interpolation space is greater than the number of interpolation data and some extra conditions can be imposed in order to guarantee a unique solution.

The emphasis is on theoretical rather than practical aspects of the problems, more precisely on discussing if their solvability depends on the choice of the set of nodes and derivatives. Most of the interesting problems are almost regular; i.e., they are solvable for almost all choices of nodes. But it is difficult to identify in advance, in a practical way, the negative choices. Very few pages are devoted to constructive approaches, and the reader should not expect a collection of methods of solution for many problems. But this is a book, written in a very clear style, that every mathematician interested in multivariate interpolation must know.

MARIANO GASCA

E. M. NIKISHIN AND V. N. SOROKIN, *Rational Approximations and Orthogonality*, Translations of Mathematical Monographs, Vol. 92, American Mathematical Society, 1991, viii + 221 pp.

The concept of rational approximation forms the background of this book. In the framework of number theory it leads to classical Diophantine approximations, while in function theory it gives rise to Padé-type approximants. For the latter the notion of orthogonality plays a central role. This approach allows one to connect the theory of Padé approximants via the theory of orthogonal polynomials with different branches of mathematics such as operator theory and mathematical physics. Some additional techniques from the theory of boundary values of analytic functions and potential theory are the advanced tools of this investigation.